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Relative Sensitivity of Conditional Distributions to Kurtosis Deviations in the Joint Model

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Abstract

The Multivariate Exponential Power family is considered for n-dimensional random variables, $\mathbf{Z}$, with a known partition $\mathbf{Z} \equiv (\mathbf{Y},\mathbf{X})$ of dimensions p and n-p, respectively. An infinitesimal variation of any parameter produces both conditional and marginal distributions perturbations. The aim of our study is to determine the local effect of kurtosis deviations by means of the Kullback-Leibler divergence measure between probability distributions. The additive decomposition of this measure in terms of the conditional and marginal distributions, $\mathbf{Y}|\mathbf{X}$ and $\mathbf{X}$, has been used to define the relative sensitivity of the conditional distributions family $\{\mathbf{Y}|\mathbf{X}=\mathbf{x}\}$. The obtained results show that, for large dimensions, it is nearly p/n.

Keywords: Multvariate Exponential Power distributions; Kurtosis; Kullback-Leibler divergence; Relative sensitivity

1. Main text

The Kullback-Leibler (KL) divergence between two elements in a parametric family of densities, $\{f(\mathbf{z},\boldsymbol{\theta});\ \boldsymbol{\theta} \in \Theta\}$, with values in n-dimensional space, is directly related to Fisher information in such a way that this can be considered as a local divergence measure, evaluating the impact of infinitesimal changes in the parameter vector $\boldsymbol{\theta}$ [e.g., S. Blyth (1994)]. If the random variable $\mathbf{Z}$ is partitioned into components $\mathbf{Y}$, of dimension p, and $\mathbf{X}$, the joint density factorization into the conditional and marginal densities, $\mathbf{Y}|\mathbf{X}$ and $\mathbf{X}$, implies a partition of the KL divergence as follows

$$D_{KL}\left(f, f^{(\Delta)}\right) = E_{\mathrm{X}}\left[D_{KL}\left(f_{Y|X}, f_{Y|X}{}^{(\Delta)}\right)\right] + D_{KL}\left(f_{\mathrm{X}}, f_{\mathrm{X}}{}^{(\Delta)}\right), \tag{1}$$

where $f^{(\Delta)}$ denotes a perturbed version of f. This decomposition immediately suggests a valuation of the relative effect of a perturbation $\Delta\boldsymbol{\theta}$ both on the family of conditionals $\{\mathbf{Y}|\mathbf{X}=\mathbf{x}\}$ and on the marginal distribution of $\mathbf{X}$. More concretely, the relative sensitivity of these structures to infinitesimal perturbations in $\boldsymbol{\theta}$ will be measured by the limit of the quotients

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$$\frac{E_{X}\left[D_{KL}\left(f_{Y|X}, f_{Y|X}{}^{(\Delta)}\right)\right]}{D_{KL}\left(f, f^{(\Delta)}\right)}, \quad \frac{D_{KL}\left(f_{X}, f_{X}{}^{(\Delta)}\right)}{D_{KL}\left(f, f^{(\Delta)}\right)}, \tag{2}$$

respectively.

We focus on a class that is of particular interest in applications because it generalizes the multivariate normal. The family MEP(μ, Σ, β)—Multivariate Exponential Power, where $\mu \in \Re^{n}$, Σ is a positive definite matrix and $\beta \in (0,1)$ is the normality parameter. This parameter provides a characterization of the tail weight of the distribution; $\beta=1$ corresponds to the *n*-dimensional multivariate normal. The parameters μ and Σ play a similar role to their counterparts in the normal case [e.g., Gómez, E. et al. (1998)]. In these conditions, the aim of our study is to calculate the relative sensitivity of this family to β perturbations around $\beta=1$.

In Main, P. et al. (2009), the KL divergence between conditional distributions is determined; there, evaluating $E_{X}\left[D_{KL}\left(f_{Y|X}, f_{Y|X}{}^{(\Delta)}\right)\right]$ yields

$$\log \frac{\int_0^\infty t^{(p/2)-1} \exp\{-\frac{1}{2}(t+q_x)^\beta dt}{2^{p/2}\Gamma(p/2)} - \frac{1}{2}\left[p - \frac{q_x{}^{\beta+p/2}}{2^{p/2}} U(a=p/2, b=\beta+p/2+1, s=q_x/2) \right], \tag{3}$$

where *U(a,b,s)* is the *Confluent Hypergeometric* function and q_x is distributed as a χ^2 with *n-p* degrees of freedom. Combining (3) and the result of the divergence between the joint distributions, which gives

$$D_{KL}\left(f, f^{(\Delta)}\right) = \log \frac{2^{\frac{n}{2\beta}}\Gamma\left(\frac{n}{2\beta}\right)}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\beta} - \frac{1}{2}\left(n - \frac{2^{\beta}\Gamma\left(\frac{n}{2}+\beta\right)}{\Gamma\left(\frac{n}{2}\right)} \right), \tag{4}$$

the quotient of interest is completely specified. Both, Monte Carlo methods of simulation and the approximation, around $\beta = 1$, by a linear function, lead us to the value *k* for the relative sensitivity of the conditionals as $n,p \to \infty$ restricted to $p/n = k$.

2. References

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